Effective toughness of heterogeneous brittle materials

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Abstract

A heterogeneous brittle material characterized by a random field of local toughness $K_c(\boldsymbol{x})$ can be represented by an equivalent homogeneous medium of toughness, K_{eff}. Homogenization refers to a process of estimating K_{eff} from the local field $K_c(\boldsymbol{x})$. An approach based on a perturbative expansion of the stress intensity factor along a rough crack front shows the occurrence of different regimes depending on the correlation length of the local toughness field in the direction of crack propagation. A *"weak pinning"* regime takes place for long correlation lengths, where the effective toughness is the average of the local toughness. For shorter correlation lengths, a transition to *"strong pinning"* occurs leading to a much higher effective toughness, and characterized by a propagation regime consisting in jumps between pinning configurations.

Key words: brittleness, homogenization, crack arrest

1 Introduction

Brittle materials such as ceramics, glass and rocks are known to be extremely sensitive to bulk and surface defects, from which cracks can be initiated eventually leading to failure. This extreme sensitivity calls for a statistical analysis of crack initiation, which has been extensively developed following the pioneering work of Weibull (Weibull 1939; Freudenthal 1968). This approach, describing the inception of crack propagation caused by initial defects (Jayatilaka and Trustrum 1977), has been progressively extended to account for various statistical distributions of bulk or surface

defects (Munz and Fett 1999), multiaxial criteria for critical loads on defects (Batdorf and Heinish 1978; Evans 1978), inhomogeneous stress fields (Davies 1973). In this "weakest-link" approach, the analysis is focused on initiation, implicitly assuming that the propagation stage is obtained systematically over unlimited distances, due to the lower loading needed for this stage as compared to the initiation one.

However, in some cases, even though cracks have been nucleated, they will not propagate to large distances, and their presence may still be acceptable in service condition of a given structure. Simple examples of such confined cracks are those induced by indentation (Lawn 1993). In the latter case, the stress field has a rapid decay with distance from the indentation point, and hence a crack which can easily be nucleated may stop shortly after initiation. In ceramic / metal assemblies, residual stresses caused by the coefficient of thermal expansion mismatch can also prevent cracks to traverse the brittle part so that a weakest link hypothesis does not apply (Charles and Hild 2002). In brittle-matrix composites, crack arrest is also observed due to the bridging forces induced by the fibers (Evans 1990).

Such situations require the characterization of the conditions under which a "macroscopic" crack may or may not propagate. The word "macroscopic" requires a specific attention. At a microscopic scale, λ, a number of non-linear phenomena may take place in the so-called process-zone (*e.g.,* dislocation emission, damage initiation, or simply non-linear debonding). However, at a larger scale, the effect of all those confined non-linearities can be characterized by a toughness K_c which will dictate whether a crack of stress intensity factor K can (when $K \geq K_c$) or cannot (when $K < K_c$) propagate. The situation considered in the present study is when, at a scale larger than λ, the toughness may vary from places to places, $K_c(\boldsymbol{x})$ thus being a random field whose characteristics are:

- a probability distribution function $p(K_c)$;
- a correlation function C defined as

$$C(\boldsymbol{x}) = \langle K_c(\boldsymbol{y}) K_c(\boldsymbol{x}+\boldsymbol{y}) \rangle_{\boldsymbol{y}} - \langle K_c(\boldsymbol{y}) \rangle_{\boldsymbol{y}}^2 \,, \tag{1}$$

  where the brackets $\langle ... \rangle_{\boldsymbol{y}}$ denote an average over the coordinate $\boldsymbol{y}$.

Let us focus in the following on cases where the correlation length ξ (above which $C(x) \approx 0$) can be defined, at least along the crack front. The term "macroscopic" refers specifically to the case of a crack whose front length L is much greater than ξ_x along the direction of the crack front. In such a situation, the medium can be characterized by an effective toughness that controls the propagation or arrest of the crack. More precisely, the local crack front roughness caused by the random toughness landscape can be ignored,

and an equivalent crack having a straight front and the same mean position is defined. Along this equivalent crack, a loading leading to a constant stress intensity factor (SIF) can be considered and the same loading will be applied simultaneously on the heterogeneous material. The critical SIF computed on the equivalent geometry which corresponds to the onset of propagation will thus characterize the effective toughness, K_{eff}.

The problem of a crack propagating in a heterogeneous toughness field has been considered both numerically and experimentally, for two-phase materials having well defined geometries. In the context of a layered toughness along the mean crack front, Eriksson (Eriksson 1998) used an energy based argument to estimate the effective toughness by a rule of mixture. The bypassing of a tough inclusion has been simulated numerically in great details (Bower and Ortiz 1991; Bower and Ortiz 1993) and corresponding experiments have been performed (Mower and Argon 1978). Curtin (Curtin 1997; Curtin 1998) performed a simplified analysis of a similar problem. To analyze the crack advance in fiber reinforced composites, instead of toughness distributions, strength distributions have been used (Beyerlein et al. 1997a; Beyerlein et al. 1997b; Curtin 1998; Landis et al. 2000). However the question of the large scale effective toughness was not considered. The relaxation of a crack distorsion due to an obstacle in a dynamic situation has also been considered in (Morrissey and Rice 2000; Rice 2001; Woolfries and Willis 1999), with a statistical treatment addressing the question of crack front roughening.

The heterogeneity of the toughness field induces perturbations of the crack front geometry. A large amount of work has been carried out in recent years to estimate the effect of a rough crack geometry on the stress intensity factor at the crack front in different cases (planar or three-dimensional, static or dynamic, ...) (Gao and Rice 1989; Movchan and Willis 1995; Willis and Movchan 1995; Movchan et al. 1998). In the restricted context of a crack propagating in anti-plane geometry (mode III), Vandembroucq and Roux (Vandembroucq and Roux 1997) performed a second order expansion of the stress intensity factor in crack roughness. The second order term was shown to systematically decrease the stress intensity factor compared with the expected value for a straight crack. This was interpreted as a "strengthening" effect, *i.e.,* an increase of the apparent toughness of the material.

Alternatively, a statistical physics approach of crack front pinning by a random field is proposed by different authors mostly from a theoretical perspective (Perrin and Rice 1994; Schmittbuhl et al. 1995; Bouchaud et al. 1993; Ramanathan and Fisher 1997; Ramanathan and Fisher 1998; Tanguy et al. 1998; Bouchaud et al. 2000; Hansen and Schmittbuhl 2003) accom-
panied by some experimental investigations (Schmittbuhl and Måløy 1997).

Most of these studies, however, focus on the statistical features of crack front roughness, and interesting features attached to the onset of crack propagation interpreted as a depinning transition. However, little attention has been paid to the quantitative estimate of the effective toughness. In (Skoe et al. 2002), the question of the statistical distribution of the macroscopic SIF at the onset of propagation is addressed, again revealing a universal critical behavior close to an effective threshold. However, the quantitative value of the latter was not treated.

Crack arrest conditions have also been studied when the crack tip traverses a medium with random toughness. The problem to solve concerns a solid medium consisting of elastic brittle grains or potential arrest sites. The grain size is considered to be of constant size and toughness is constant in each grain. The toughness varies from grain to grain so that no spatial correlations on scales larger than the grain size are assumed (Chudnovsky and Kunin 1987). The same hypothesis was used by Charles et al. (Charles and Hild 2002; Charles et al. 2003) to analyze instantaneous and delayed crack propagation and arrest. Jeulin (Jeulin 1994) proposed a model in which the microstructure is assumed to be described by a Poisson mosaic. A Poisson tessellation defines the grain boundaries. The latter are made of Poisson lines in the plane for a two-dimensional medium. Instantaneous propagation and arrest conditions are investigated.

In the following, the case of a layered local toughness, invariant along the direction of propagation, is considered first, since this particular texture allows for a closed-form answer. Then slow modulations of the toughness along the propagation direction are considered, in the so-called "weak pinning" regime. Finally, the "strong pinning" case is addressed when the correlation length of the toughness perpendicular to the front becomes small. In all these cases, an emphasis is put on an estimate of the effective toughness and comments are made on specific features expected in the propagation regime.

2 Layered toughness

A crack front parallel to the x-axis is considered, with a toughness field translationally invariant along the propagation direction y, $\partial K_c(x, y)/\partial y = 0$. Starting from a straight front, as the loading is increased, the crack front begins to develop some roughness, being still pinned in some regions and propagating in other parts. The front is characterized by its coordinate along the y axis: $h(x)$. For simplicity, we assume that the crack remains planar, confined into the (x, y)-plane.

The roughness of the crack front will induce a modulation of the local SIF,

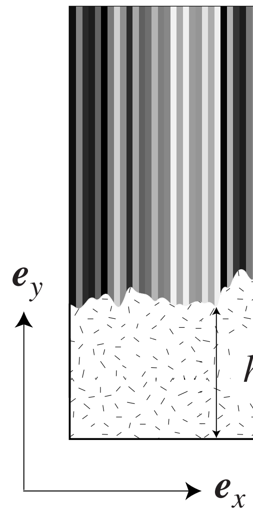

Fig. 1. Crack propagation in a layered toughness configuration. The crack front is distorted such that at every point x along the front the stress intensity factor $K(x)$ matches exactly the local toughness value $K_c(x)$.

$K(x)$, when compared to the macroscopic one K_0, defined with the same loading and a straight crack having the same mean position. By using a perturbation analysis (Gao and Rice 1989), a first order solution can be derived

$$K(x) = K_0 \left(1 + \frac{1}{\pi} \int \frac{h(x') - h(x)}{(x' - x)^2} dx' \right) . \tag{2}$$

At the onset of propagation, the stress intensity factor has to match the local toughness (*i.e.*, $K(x) = K_c(x)$). This can be achieved through a particular conformation of the crack front, $h^*(x)$, which can be obtained from a mere Fourier transform of Eq. (2). By denoting Fourier transforms with a $\tilde{}$ sign,

$$\widetilde{h^*}(k) = \frac{1}{|k|} \frac{\widetilde{K_c}(k)}{K_0} . \tag{3}$$

The integration of Eq. (2) over x, using $K(x) = K_c(x)$, allows one to derive the effective toughness with $K_{\mathrm{eff}} = K_0$

$$K_{\mathrm{eff}} = \langle K_c \rangle . \tag{4}$$

Equation (4) is an exact result, however limited to the case of a narrow distribution of local toughness so that the first order expansion (2) of the SIF modulation remains valid. We note that this simple conclusion is a general result when using a first order coupling term between different positions along the front. We note that, still using a similar linear kernel, a following section will show that a regime (termed "strong-pinning") may appear where such a simple conclusion breaks down.

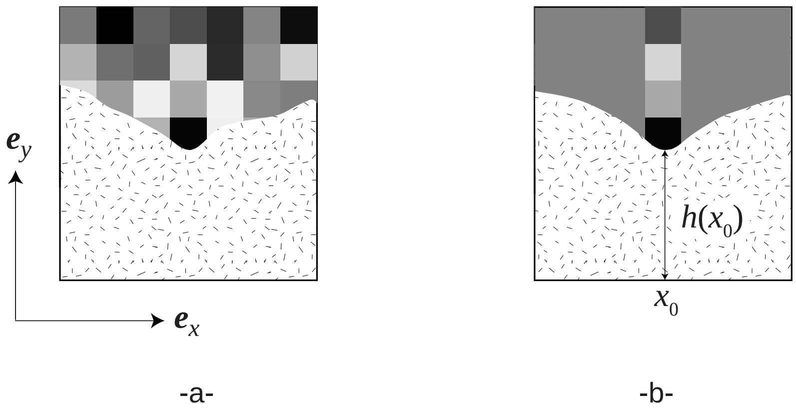

Fig. 2. A random toughness configuration is represented schematically on the left (a) with a grey-scale coded toughness. On the right, (b), a reference case is constructed by extracting the toughness pattern along a line parallel to the propagation direction, and substituting to the rest a uniform toughness environment. The latter is determined such that the deflection $\Delta h = h(x_o) - \langle h(x) \rangle_x$ is zero on average over all accessible crack front conformations.

3 Self-consistent homogenization

Let us now consider the more general case of a random toughness field, varying along the propagation direction. In contrast to the layered case, an exact result is not derived, but rather an approach based on a self-consistent approximation is proposed, similar to the one used for homogenization of a randomly heterogeneous elastic solid (Kroener 1967).

A narrow strip is considered parallel to the propagation direction y, centered on x_0, and of width ξ_x equal to the correlation length of the toughness field in the x-direction. Within this strip, the random toughness is preserved. Outside it, one substitutes to the random toughness a homogeneous toughness, K_0 as shown schematically in Figure 2. The local toughness contrast induces a perturbation of the equilibrium position of the crack front. Depending on the sign of the toughness contrast, the position $h(x_0)$ of the front at the center of the strip is either in front of or behind the average front position $\langle h(x) \rangle_x$. The value of this deflection $\Delta h(x_0) = h(x_0) - \langle h(x) \rangle_x$ is at first order directly proportional to the toughness difference $\Delta K(x_0) = K(x_0) - K_0$. In the process of homogenization, the value of K_0 is chosen such that $\langle \Delta h(x_0) \rangle_x$ vanishes, *i.e.,* the front shape averaged over all crack front conformations along the disordered strip is flat.

Such an approach turns out to provide rather accurate estimates in the case of disordered elastic solids (Willis 1991), albeit there is no way to estimate *a priori* the accuracy of the result.

Equation (2) can be used to compute the crack front shape. The stress intensity

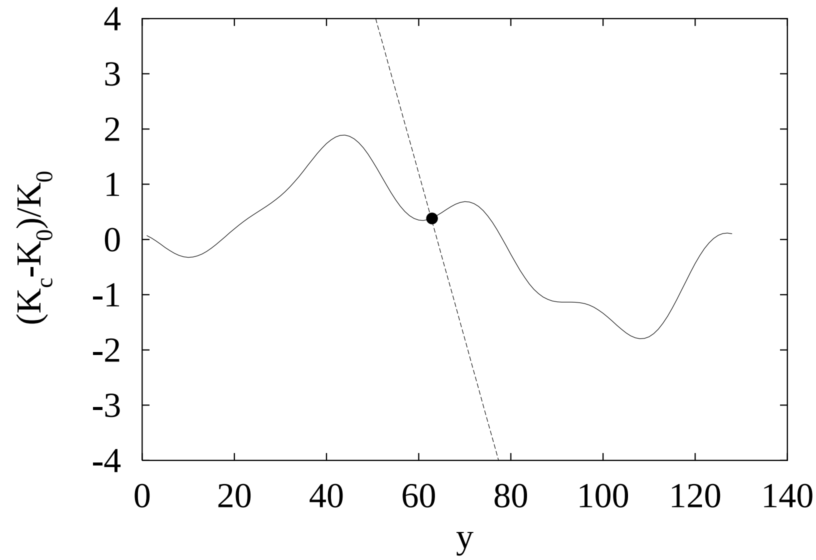

Fig. 3. Schematic construction of the equilibrium position of the crack front shape: the intersection of the toughness profile shown as a continuous curve, with the coupling due to the distorsion of the crack front shown as a dotted line gives here a unique equilibrium position shown as a black dot.

factor to consider is $K = K_0$ all along the front except in the strip where $K = K(x_0, h(x_0))$. Inversion of Eq. (3) leads to a logarithmic front shape, and thus the extension of the front along the x-direction, L, always comes into play in the difference $\Delta h = h(x_0) - \langle h \rangle$. Homogeneity also dictates that the system size has to be scaled by the strip width, ξ_x. Because of linearity, Δh is proportional to $K(x_0, h(x_0)) - K_0$ so that

$$K(x_0, \langle h \rangle + \Delta h) - K_0 = -\frac{AK_0}{\xi_x \log(L/\xi_x)} \Delta h \,, \tag{5}$$

where A is a numerical constant.

Using the linear dependence between the stress intensity factor and the distance between the crack front within the strip and away from it, the equilibrium front position can be determined as shown in Fig. 3, by plotting $K_c(x_0, \langle h \rangle + \Delta h)$ as a function of Δh together with the linear relation (see Eq. (5)).

4 Weak pinning regime

When the correlation length along the y-direction is long, so that the gradient $\partial K(x,y)/\partial y$ remains always smaller than the "stiffness" $S = AK_0/[\xi_x \log(L/\xi_x)]$, a regime referred to as "weak pinning" occurs. In this

case, there is a unique equilibrium position for the crack front (Fig. 3) since the "stiffness", S, is greater than the gradient of toughness in the propagation (y-)direction. To compute the effective toughness $K_{\text{eff}} = K_0$, one has to integrate the K_c values with a measure corresponding to a uniform sampling of $\langle h \rangle$. This introduces a bias in the weighting of the distribution of K_c values

$$
\begin{aligned}
K_{\text{eff}} &= \lim_{Y\to\infty} \frac{1}{Y} \int_0^Y K_c(y) \frac{d(\Delta h)}{dy} dy \\
&= \lim_{Y\to\infty} \frac{1}{Y} \int_0^Y K_c(y) \left(1 + \frac{1}{S}\frac{dK_c(y)}{dy}\right) dy \,.
\end{aligned} \tag{6}
$$

The first term gives the simple arithmetic average of the local toughness, similar to the result obtained in the layered case. The second term is proportional to the correlation between the local toughness and its gradient. By performing the change of integration variable from y to K_c^2, it becomes apparent that in the case of a stationary toughness field, this second integral vanishes. Thus, in the weak-pinning regime, the same result as for the layered case applies

$$K_{\text{eff}} = \langle K_c \rangle \,. \tag{7}$$

Therefore this simple result appears to be quite robust, and gives confidence to the self-consistent approach proposed herein.

In this scenario, the propagation is smooth, *i.e.,* the local crack front advance is a continuous function of the mean crack position. The final result concerning the effective toughness could also have been derived using different approaches, such as a global energy balance (Eriksson 1998), resorting to the energy release rate rather that the crack toughness. We however refrain from using such a treatment because of the following section, where this argument will be shown to break down.

5 Strong pinning regime

From the simple geometric construction leading to the equilibrium position of the crack front, one sees that a very different behavior appears as soon as the "stiffness", S, becomes smaller than the gradient of toughness in the propagation (y-) direction. In this case, illustrated in Fig. 4, multiple solutions can be found. In generic cases, a sequence of alternatively stable and unstable solutions is obtained. However, even ignoring those unstable positions, the multiplicity of solutions indicates that the position will be selected through the history of propagation. In the case of interest, the mean crack front position is chosen to be monotonically increasing, so that the toughness values

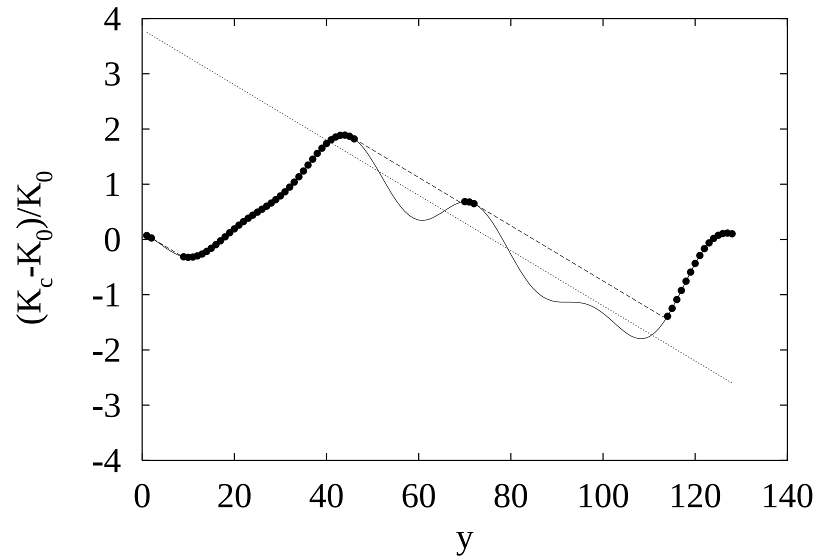

Fig. 4. Schematic construction for the strong pinning case. The continuous curve shows the local toughness profile. In contrast to the previous figure, many solutions of equilibrium position exists as shown by the five intersections of the dotted line with the continuous curve (three stable positions and two unstable ones). This selects only a subset of accessible arrest sites shown by bold dots, separated by sudden jumps shown by a dashed line of slope $-S$.

sampled in the strip are confined to the highest values, as if the toughness profile were illuminated by a grazing incidence light (*i.e.,* exposed hill tops correspond to sampled equilibrium position, whereas shadowed regions correspond to unstable jumps to a new arrest position). In contrast to the weak-pinning case where the propagation was smooth, one observes in the strong-pinning case a sort of stick-slip crack propagation. This regime is the one addressed in the statistical approaches developed in (Bouchaud et al. 1993; Ramanathan and Fisher 1997; Ramanathan and Fisher 1998; Tanguy et al. 1998; Bouchaud et al. 2000; Hansen and Schmittbuhl 2003).

To estimate K_{eff}, it is necessary to evaluate the probability that a particular value of the toughness $K_c(x)$ can constitute an accessible equilibrium position. This implies that for all $x' < x$, $K_c(x') < K_c(x) + S(x - x')$. To proceed, it is convenient to specialize the toughness profile to a simple pattern, such as a piecewise constant toughness over intervals of equal size ξ_y (chosen equal to correlation length along the y direction), and without correlation from one interval to the next, as shown schematically in Figure (2a).

Let $P(K_c)$ be the cumulative toughness distribution $P(K) = \int_0^K p(k)dk$. The

probability, $f(K_c)$, that K_c can be reached is written

$$
\begin{aligned}
f(K_c) &= p(K_c) \prod_{i=1}^{\infty} P(K_c + iS\xi_y) \\
&\approx p(K_c) \exp\left(1/(S\xi_y) \int_0^{\infty} \log(P(K_c + x))dx\right) .
\end{aligned}
\tag{8}
$$

This distribution is not normalized, as can be seen by the existence of shadow zones containing unaccessible sites. Thus, the p.d.f. of effective stable and accessible positions is

$$
\phi(K_c) = \frac{f(K_c)}{\int_0^{\infty} f(k)dk} \tag{9}
$$

and thus the effective toughness is again the simple arithmetic average of K_c using the probability distribution ϕ instead of p

$$
K_{\text{eff}} = \frac{\int_0^{\infty} f(k)k\ dk}{\int_0^{\infty} f(k)\ dk} . \tag{10}
$$

This last equation, together with the definition of f in Eq. (8), gives an estimate of the effective toughness based on the self-consistent approach. Even though it was derived in the context of strong-pinning, in the weak pinning regime, Eq. (8) provides $f(k) = p(k)$, and hence the previously derived result can be recovered.

It is worth noting that the correlations along the crack front and perpendicular to it contribute significantly to the final formula. Consequently, the effective toughness is not a specific property of the material along the potential crack plane, but may depend on the direction of crack propagation. This result contrasts with energy-dissipation based arguments where such an orientation effect cannot appear. The fundamental basis for such a difference comes from the specificity of the strong pinning regime, namely the occurrence of sudden jumps over unstable configurations. During those jumps, the potential energy is transferred to kinetic energy, and this dynamic aspect is not considered (*i.e.,* the implicit assumption is that most of this kinetic energy is either dissipated or radiated away from the crack so that it is no longer available for propagation). This effect explains why energy-based approaches will not reproduce the proposed result. Furthermore, the crack length comes into play in the expression of K_{eff} in the strong-pinning regime through the particular dependence of the stress intensity factor on the crack front distorsion. However, this dependence is only logarithmic and might be difficult to observe.

Finally, the emphasis put on the multistability of the strong pinning regime is comparable to parallel analyses performed in other contexts such as solid friction (Caroli and Nozières 1996), wetting phenomena (hysteresis of wetting angle) (Adamson and Gast 1998), or charge density waves (Fisher 1985) where strong pinning is at the origin of new phenomena at the macroscopic scale as

compared to the microscopic one, and where equilibrium concepts and energy balance arguments are no longer operational.

6 Summary

In this article, the homogenization of a random toughness field is addressed, taking into account the effect of the crack front roughness on the local stress intensity factor to first order in perturbation. Starting from a layered case, where it can be shown that the effective toughness is equal to the arithmetic average of the toughness, a self-consistent scheme is introduced to deal with more general toughness fields. This approach allows one to extend the validity of the effective toughness estimate to slowly varying toughness fields along the propagation direction, a situation referred to as weak-pinning. However, as the correlation length of the toughness field along the propagation direction decreases, a novel behavior is encountered when the crack front position is no longer single-valued. In the so-called strong-pinning regime, the sampling of the local toughness becomes inhomogeneous, and gives rise to an apparent strengthening. The transition between these two regimes is progressive, and is accompanied at the microscopic level by an unsteady propagation displaying sudden jumps or burst similar to stick-slip.

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